Linear systems – Midterm exam

Midterm exam 2024–2025, Thursday 5 June 2025, 9:00 – 11:00

Instructions

1. The use of books and lecture notes is not allowed, but you can use a one-page cheat sheet.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1

(6 + 10 + 8 + 10 = 34 points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad \text{with} \qquad A = \begin{bmatrix} 5 & 1 & -1 \\ -14 & -3 & 3 \\ 16 & 3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}. \tag{1}$$

(a) Show that the system is not controllable.

(b) Find a nonsingular matrix T and matrices A_{11}, A_{12}, A_{22}, and B_1 such that

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

where the matrix pair (A_{11}, B_1) is controllable.

Hint. It is sufficient to give T^{-1} instead of T, but please give A_{11}, A_{12}, A_{22}, and B_1.

(c) Use the result of (b) to show that the system (1) is stabilizable.

(d) Use the matrix T from (b) to find a matrix F such that the feedback $u(t) = Fx(t)$ stabilizes the system (1).

Problem 2

(18 points)

Find all values of $a \in \mathbb{R}$ and $b \in \mathbb{R}$ such that the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -2 & -a & -1 \end{bmatrix} x(t)$$

is asymptotically stable.

Problem 3 (12 + 8 + 8 + 10 = 38 points)

Consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Gd(t), \\ y(t) &= Cx(t), \end{aligned} \tag{2}$$

with state $x(t) \in \mathbb{R}^n$, external disturbance $d(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}^p$. For initial condition $x(0) = x_0$ and disturbance $d : [0, \infty) \to \mathbb{R}^m$, we denote by $y(t; x_0, d)$ the corresponding output solution, i.e.,

$$y(t; x_0, d) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Gd(\tau)\,\mathrm{d}\tau.$$

We say that the system (2) is *disturbance decoupled* if the disturbance d does not affect the output solution, i.e., $y(t, x_0, d_1) = y(t, x_0, d_2)$ for all $t \geq 0$, for all $x_0 \in \mathbb{R}^n$ and all d_1, d_2.

(a) Show that the system is disturbance decoupled if and only if

$$Ce^{At}G = 0 \text{ for all } t \geq 0. \tag{3}$$

Next, consider the following statements:

(i) $CA^kG = 0$ for $k = 0, 1, 2, \ldots$.

(ii) there exists an A-invariant subspace $\mathcal{V} \subset \mathbb{R}^n$ such that $\operatorname{im} G \subset \mathcal{V} \subset \ker C$.

Note that (3) is equivalent to (i), meaning that the system is disturbance decoupled if and only if (i) holds. In the remainder of this problem, we will show (i) $\iff$ (ii).

(b) Let $\mathcal{V} \subset \mathbb{R}^n$ be an A-invariant subspace satisfying $\operatorname{im} G \subset \mathcal{V}$. Show that

$$\operatorname{im} A^kG \subset \mathcal{V} \text{ for } k = 0, 1, 2, \ldots.$$

(c) Use the result from (b) to show that (ii) implies (i).

(d) Show that (i) implies (ii).

(10 points free)